Response of a Two-Body Gravity Gradient System in a Slightly Eccentric Orbit

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This analysis examines the effect of small orbit eccentricity on a gravity gradient system with a single damping boom. The effect of a small eccentricity is felt as a sinusoidal disturbance torque and leads to a cyclic pointing error. Conditions for minimum steady-state response and the influence on transient performance are investigated using conventional techniques. The conditions for optimum steady-state and transient response are found to conflict. The choice of parameters for optimum over-all system performance thus depends on the relative importance of the time to damp and the magnitude of the steady-state error. Curves are presented to aid the designer in making this choice.

Nomenclature

$A_iB_iC_i$	= moments of inertia of body i about the roll, pitch and yaw axes
a	= semimajor axis of orbit
a,b,c,d,e	= coefficients in characteristic equation
D_1	= real part of the maximum root
F	= nondimensional spring parameter $F = k/3B_2\omega^2 p_1$
G	= nondimensional damping parameter $G = \zeta/B_2(3p_1)^{1/2}$
G_i	= gravity gradient torque on ith body
K	= ratio F/F_L
k	= spring constant, ft-lb/rad
L_i	= eccentric forcing function on ith body
P,Q	= const
p_i	= inertia parameter $p_i = (A_i - C_i)/B_i$
R	= ratio p_2/p_1
s	= Laplace variable with respect to τ
T_{ij}	= torque on body i due to body j
t	= time, sec
t_e	= $1/e$ damping time, orbits
U,V,W	= const
X,Y,Z	= inertial axes
$x_iy_iz_i$	= body axes
α	= angle between main boom and local vertical, rad
β	= angle between damper boom and horizontal, rad
γ	= frequency $\gamma = 1/(3p_1)^{1/2}$ rad/sec
δ	= inertia ratio B_2/B_1
ϵ	= orbit eccentricity
ζ	= damping constant, ft-lb/rad/sec
τ	= nondimensional time $\tau = \omega t(3p_1)^{1/2}$
Φ	= performance index
ω	= orbital rate in a circular orbit, rad/sec

Subscripts

E	= extremal
L	= limiting
ss	= steady state
1,2	= main body and damper boom, respectively

Introduction

In studying gravity gradient systems for long life applications the primary performance parameter is usually the accuracy with which the local vertical can be maintained. Rapid acquisition is desirable but often of secondary importance. Reference 1 shows that acquisition time can be minimized by a particular choice of system parameters, but this does not guarantee optimum steady-state response. The present analysis was undertaken to explore the conditions governing accuracy in a slightly eccentric orbit and to determine the penalty, if any, paid in acquisition.

The full three-axis equations of motion for a gravity gradient system of the Vertistat type[1] are of sixth order. However if small angle approximations are taken it can be shown that the equations reduce to two uncoupled sets, one defining the pitch, or orbital plane motion and the other describing the coupled roll/yaw motion. In addition the effect of small orbit eccentricity is felt only in the pitch plane. Thus, a linear analysis can be conducted disregarding roll and yaw motion (and the presence of a cross-orbit damping boom) and concentrating exclusively on motion of the main body and one boom in the orbit plane.

The equations describing the simplified system are of fourth order, indicating that the response consists of at least two modes. The difficulty immediately arises of how best to define a performance criterion. Many such criteria have been proposed in the form of an integral of a function of the error or of the error and time, for example the integral of square error (ISE). In selecting criteria, the system objectives should be kept in mind. For gravity gradient systems rapid acquisition and small steady-state error are desirable. Zajac[2] has used the maximum damping of the least damped mode as a criterion which provides a useful and convenient figure of merit to assess acquisition performance. As a measure of accuracy the obvious selection of amplitude of error is indicated. This would not be possible if the system were subjected to a variety of disturbances. However by focusing attention on the effect of orbit eccentricity, which can be approximated by a sinusoidal torque, it is possible to define explicitly the amplitude of the steady-state error.

An expression for the error magnitude is derived in terms of system parameters and the conditions for minimum error determined. It is found that system parameters can be chosen to cause the error to tend to zero. Unfortunately as the error tends to zero the time to damp tends to infinity. In short, the conditions for good steady-state and good transient performance conflict. The choice of system parameters thus depends on a trade which reflects the relative importance of the two conflicting requirements.

Equations of Motion

The general arrangement is shown in Fig. 1. The system is composed of a main satellite body, distinguished by subscript 1, and a secondary damping boom, subscript 2, which is attached to the main body through a single degree-of-freedom hinge. Orbit axes XYZ are taken with the origin at the system center of mass, Z extending along the local vertical, X in the general direction of flight and Y normal to the orbit plane. These axes rotate at orbital rate, ω, with respect to inertial space about axis Y. Two sets of axes, $x_1y_1z_1$

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and $x_2y_2z_2$ are fixed in the bodies. In general, the center of masses of the two parts of the vehicle will not coincide, but it is shown in Ref. 1 that a separation is easily accommodated by a modification of the moment of inertia expressions. Little loss of generality is introduced if the centers of mass are taken as coincident. If this is assumed, then the three systems of axes have a common origin at the hinge point. Finally, planar motion only is considered, about the common Y_1, y_1 and y_2 axes. $x_1y_1z_1$ are related to XYZ by a rotation α and $x_2y_2z_2$ related to XYZ by a rotation β and represent respectively the error angle and the displacement of the boom from its nominal horizontal position. The desired equilibrium condition is $\alpha = \beta = 0$.

Body 1

Consider rotations about the center of mass. By equating the product of inertia and angular acceleration to the applied torque

$$\beta_1\ddot{\alpha} = G_1 + T_{12} + L_1 \tag{1}$$

The gravity gradient torque, for small α, is

$$G_1 = 3\omega^2(C_1 - A_1)\alpha \tag{2}$$

and the torque on body 1 due to body 2, introduced by the spring and damper restraints, can be expressed

$$T_{12} = -k(\alpha - \beta) - \zeta(\dot{\alpha} - \dot{\beta}) \tag{3}$$

Rowell and Smith[3] have shown that the forcing function caused by small orbit eccentricity can be approximated by

$$L_1 = 2B_1\epsilon\omega^2 \sin\omega t \tag{4}$$

Substituting (2), (3), and (4) into (1)

$$B_1\ddot{\alpha} + \zeta(\dot{\alpha} - \dot{\beta}) + k(\alpha - \beta) + 3\omega^2(A_1 - C_1)\alpha = 2B_1\epsilon\omega^2 \sin\omega t \tag{5}$$

Body 2

By the method just shown

$$B_2\beta = G_2 + T_{21} + L_2 \tag{6}$$

Also

$$G_2 = 3\omega^2(C_2 - A_2)\beta, \quad T_{12} = -T_{21} \tag{7}$$

$$L_2 = 2B_2\epsilon\omega^2 \sin\omega t \tag{8}$$

Combination leads to

$$B_2\ddot{\beta} - \zeta(\dot{\alpha} - \dot{\beta}) - k(\alpha - \beta) + 3\omega^2(A_2 - C_2)\beta = 2B_2\epsilon\omega^2 \sin\omega t \tag{9}$$

Equations (5) and (9) are the basic equations of motion with α and β as the unknowns and the moments of inertia, the damping and hinge constants and the eccentricity as system parameters. It will be convenient to nondimensionalize the equations in order to simplify the expressions and to consolidate the system parameters. This is accomplished by introducing (see Nomenclature) the inertia parameters p_1 and p_2 for bodies 1 and 2, the spring and damping parameters F and G, $R = p_2/p_1$, $\delta = B_2/B_1$, nondimensional time $\tau = \omega t(3p_1)^{1/2}$, and the frequency parameter $\gamma = (\frac{1}{3}p_1)^{1/2}$. Then Eqs. (5) and (9) become

$$s^2\bar{\alpha} + \delta(Gs + F)(\bar{\alpha} - \bar{\beta}) + \bar{\alpha} = 2\epsilon\gamma^3/(s^2 + \gamma^2) + \dot{\alpha}_0 + s\alpha_0 + \delta G(\alpha_0 - \beta_0) \tag{10}$$

$$s^2\bar{\beta} - (Gs + F)(\bar{\alpha} - \bar{\beta}) + R\bar{\beta} = 2\epsilon\gamma^3/(s^2 + \gamma^2) + \dot{\beta}_0 + s\beta_0 - G(\alpha_0 - \beta_0) \tag{11}$$

Here s represents the Laplace transform with respect to τ.

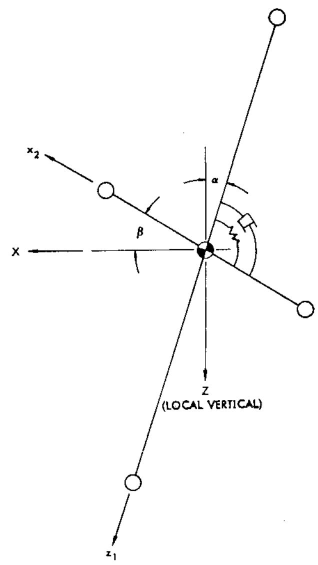

Fig. 1. General arrangement of gravity gradient system.

Use has been made of the relations

$$\mathcal{L}\dot{\theta} = \omega s(3p_1)^{1/2}, \quad \mathcal{L}\ddot{\theta} = 3s^2\omega^2 p_1 \tag{12}$$

which follow directly from the definition of τ. From (11)

$$\bar{\beta} = [2\epsilon\gamma^3/(s^2 + \gamma^2) + (Gs + F)\bar{\alpha} + \dot{\beta}_0 + s\beta_0 - G(\alpha_0 - \beta_0)][1/(s^2 + Gs + F + R)] \tag{13}$$

Substitution of (13) into (10) leads to the solution for $\bar{\alpha}$:

$$[C.E]\bar{\alpha} = 2\epsilon\gamma^3/(s^2 + \gamma^2)[s^2 + (Gs + F)(1 + \delta) + R] + \dot{\alpha}_0(s^2 + Gs + F + R) + \alpha_0[s^3 + Gs^2(1 + \delta) + s(F + R) + \delta GR] + \delta\dot{\beta}_0(Gs + F) + \delta\beta_0(Fs - GR) \tag{14}$$

$[C.E]$ is the characteristic equation of the system

$$[C.E] = as^4 + bGs^3 + cs^2 + dGs + e \tag{15}$$

where

$$a = 1, \quad b = 1 + \delta \tag{16}$$

$$c = 1 + R + (1 + \delta)F, \quad d = 1 + \delta R \tag{17}$$

$$e = R + (1 + \delta R)F \tag{18}$$

Equation (14) is the basic response equation on which all manipulations will be made. It is seen that the expression contains the four system parameters F, G, δ, and R. F and G describe the damper characteristics, while δ and R are functions of the configuration. Implied, but not directly visible, are p_1 and p_2, which are also functions of the configuration.

The problem reduces to an examination of the system response as affected by the hinge parameters and the vehicle configuration. In particular it is desired to minimize the amplitude of the steady-state response by a suitable choice of the system parameters.

Considerable simplification can be achieved by fixing, at the outset, the values of p_1 and p_2. Suppose $p_1 = 1$ and $p_2 = -1$. The expression $p_2 = -1$ means that the damper boom is an ideal dumbbell ($B_2 = C_2$, $A_2 = 0$). All existing damper booms are good approximations of dumbbells and there is no reason to suppose future booms will be any different. The expression $p_1 = 1$ means either the main body is a dumbbell, or a dumbbell with the addition of a transverse body normal to the orbit plane (for example a roll/yaw damper boom). The equivalence of the alternate configurations is easily proved; suppose the transverse boom

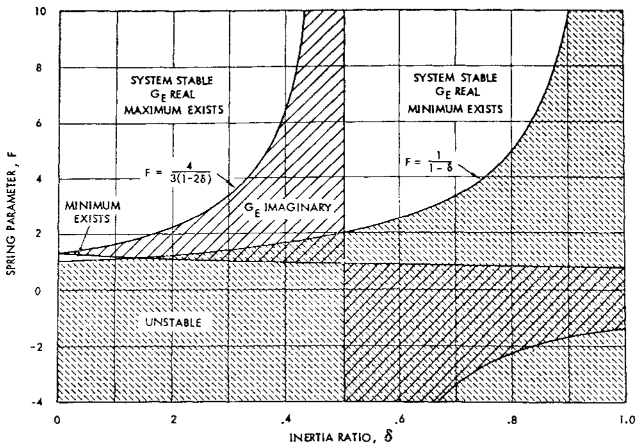

Fig. 2 Stability and realizability boundaries.

has moments of inertia $A_3 = B_3$, $C_3 = 0$ then the effective p_1 is

$$p'_1 = (A_1 + A_3 - C_1 - C_3)/(B_1 + B_3 = (A_1 - C_1)/B_1 = p_1 \quad (19)$$

The presence of inertia augmentation booms will ensure that A_1 and B_1 are large compared to C_1. Thus $p_1 = 1$ will be closely approximated in the majority of practical configurations. If the $p_1 = 1$ and $p_2 = -1$ assumptions are made it follows that $R = -1$ and the system parameters reduce to three: F, G, and δ. F and G define the hinge spring and damping characteristics while δ is a measure of the relative size of the damping boom.

Response Under Eccentricity

From Eq. (14) the response of the system in the frequency domain can be expressed

$$\bar{\alpha} = 2\epsilon\gamma^3[s^2 + (Gs + F)(1 + \delta) - 1]/[(C.E)(s^2 + \gamma^2)] + f(s)/(C.E) \quad (20)$$

where $f(s)$ is a function entirely of the initial conditions,

$$f(s) = s^3\alpha_0 + s^2[G(1 + \delta)\alpha_0 + \dot{\alpha}_0] + s[(F - 1)\alpha_0 + G\dot{\alpha}_0 + \delta G\dot{\beta}_0 + F\delta\beta_0] + (F - 1)\dot{\alpha}_0 - \delta G(\alpha_0 - \beta_0) + F\delta\beta_0 \quad (21)$$

By the method of partial fractions, Eq. (20) may be expressed in the form

$$\bar{\alpha} = (k_1 s + k_2\gamma)/(s^2 + \gamma^2) + g(s)/(C.E) \quad (22)$$

If the system is stable the two terms in (22) represent the steady-state solution and the transient response respectively. The conditions for stability can be found by applying Routh's criteria to the characteristic equation, (15), that is

$$a, bG, c, dG, e > 0; \; d(bc - ad) > b^2e \quad (23)$$

For the system under consideration these conditions reduce to

$$G > 0, \quad \delta < 1, \quad F > 1/(1 - \delta) \quad (24)$$

Provided these conditions are met the steady-state solution from (22) is

$$\alpha = k_1 \cos\gamma\tau + k_2 \sin\gamma\tau \quad (25)$$

To evaluate k_1 and k_2 the right-hand sides of Eqs. (20) and (22) are equated. Multiplying through by $(s^2 + \gamma^2)$ and setting $s = i\gamma$ reduces the expression to

$$2\epsilon\gamma^3[-\gamma^2 + (F + i\gamma G)(1 + \delta) - 1]/[C.E]_{s=i\gamma} = i\gamma k_1 + \gamma k_2 \quad (26)$$

Evaluation and separation into real and imaginary parts yields expressions for k_1 and k_2

$$k_1 = 2\epsilon(UQ - PV)/3(U^2 + V^2) \quad (27)$$

$$k_2 = 2\epsilon(UP + VQ)/3(U^2 + V^2) \quad (28)$$

Where

$$P = F(1 + \delta) - \tfrac{4}{3} \quad (29)$$

$$Q = G(1 + \delta)/(3)^{1/2} \quad (30)$$

$$U = 2[F(1 - 2\delta) - \tfrac{4}{3}]/3 \quad (31)$$

$$V = 2G(1 - 2\delta)/3(3)^{1/2} \quad (32)$$

Equation (25) shows that the response is composed of the sum of a sine and a cosine term, both of the same frequency. Since the two components are 90° out of phase it follows that the magnitude of the steady-state response is

$$|\alpha| = \alpha_{ss} = (k_1^2 + k_2^2)^{1/2} \quad (33)$$

Substitution of (27) and (28) into (33) leads to

$$\alpha_{ss} = 2\epsilon[(P^2 + Q^2)/(U^2 + V^2)]^{1/2}/3 \quad (34)$$

By a similar process

$$|\beta| = \beta_{ss} = 2\epsilon[W^2 + Q^2)/(U^2 + V^2)]^{1/2}/3 \quad (35)$$

where

$$W = F(1 + \delta) + \tfrac{2}{3} \quad (36)$$

Examination of (34) and (29–32) shows that if F is chosen such that

$$F = \tfrac{4}{3}(1 + \delta) \quad (37)$$

the steady-state error will tend to zero as G tends to zero. If the system is to remain stable however there are limitations on the permissible values of F. Using (24), it follows that F can only take the value given in (37) for a stable system if $\delta \leqq \frac{1}{7}$. Thus only for values of $\delta \leqq \frac{1}{7}$ will the steady-state error tend to zero as G tends to zero.

Conditions for Minimum Response

Some insight into the behavior of the system can be obtained by taking the conventional approach to finding extremal values. If (33) is differentiated with respect to F and rearranged

$$(\mathcal{G}/\epsilon)^2(\partial\alpha_{ss}^2)/\partial F = \delta(1 + \delta)(1 - 2\delta) \times [G^2 - \{3F - 4/(1 + \delta)\}\{3F - 4/(1 - 2\delta)\}] \quad (38)$$

where

$$\mathcal{G} = [3F(1 - 2\delta) - 4]^2 + G^2(1 - 2\delta)^2 \quad (39)$$

Extrema are thus reached when G and F (now signified by

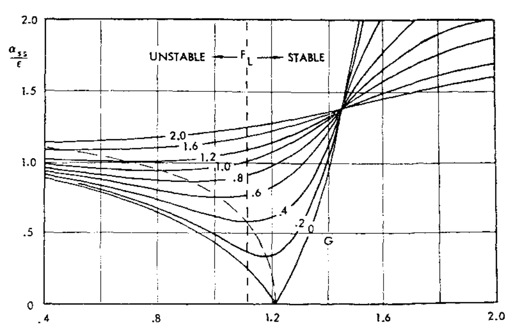

Fig. 3 Performance, $\delta = 0.1$.

subscript $_E$) are related by

$$3G_E^2 = [3F_E - 4/(1 - 2\delta)][3F_E - 4/(1 + \delta)] \quad (40)$$

Now G_E must be real, and this can only occur if

$$3F_E > 4/(1 - 2\delta) \text{ and } 3F_E > 4/(1 + \delta) \quad (41)$$

or

$$3F_E < 4/(1 - 2\delta) \text{ and } 3F_E < 4/(1 + \delta) \quad (42)$$

Also, for the system to be stable, from (24),

$$F > F_L = 1/(1 - \delta) \quad (43)$$

Combining conditions (41–43), regions can be defined in which both G_E is real and the system stable. These regions are shown in Fig. 2.

Further examination shows that the extrema expressed by (41) and (42) represent both a maximum and a minimum for $\delta < 0.5$. The conditions are

$$\begin{aligned} &\alpha_{ss}/\epsilon \text{ is a minimum for} \quad 0 \leq 3F \leq 4/(1 + \delta) \\ &\alpha_{ss}/\epsilon \text{ is a maximum for} \quad 3F \leq 4/(1 - 2\delta) \end{aligned} \quad (44)$$

For $\delta > 0.5$ there is only one extremum for real values of G_E, at $3F \geq 4(1 + \delta)$, and this is a minimum. The nature of the extrema were confirmed by plotting α_{ss}/ϵ vs F for various values of δ. Two of these curves are shown in Figs. 3 and 4 for $\delta = 0.1$ and $\delta = 0.6$.

To summarize, two regions only exist in which a minimum occurs: the triangular region bounded by $\delta = 0$, $F = F_L$, and $3F = 4/(1 + \delta)$ and the region $\delta \geq 0.5$, $F \geq F_L$. Between $\delta = \frac{1}{7}$ and $\delta = \frac{1}{2}$ no minimum exists.

The minimum is a mathematical minimum only, i.e., a point of zero slope. When the stability condition $F \geq F_L$ is imposed, the true minimum often occurs on the $F = F_L$ boundary. This is illustrated in Figs. 3 and 4.

An interesting and more useful fact emerges from further examination. If (33) is differentiated with respect to G, $\partial\alpha_{ss}^2/\epsilon^2\partial G$ is always positive, thus for any value of δ and F the amplitude of the error will decrease with a decrease in G, for all $G \geq 0.0$. The minimum response then occurs at the minimum acceptable value of G. This is illustrated in Fig. 5 which shows the amplitude of the steady-state error plotted as a function of δ with G as a parameter. The curves are drawn for the particular value of $F = 1.2F_L$ but are typical of the form exhibited for other values of F. It is seen that zero amplitude can be approached in the limit as G tends to zero for a particular value of δ. This value of δ is

$$\delta = (4 - 3F)/3F \quad (45)$$

and exists only for $0 \leq \delta \leq \frac{1}{7}$.

As G tends to zero, however, the system becomes more nearly undamped and it can be expected that transient damping performance will suffer. It appears at this point, and will

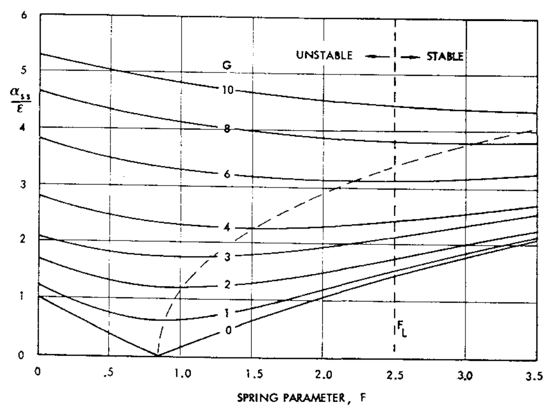

Fig. 4 Performance, $\delta = 0.6$.

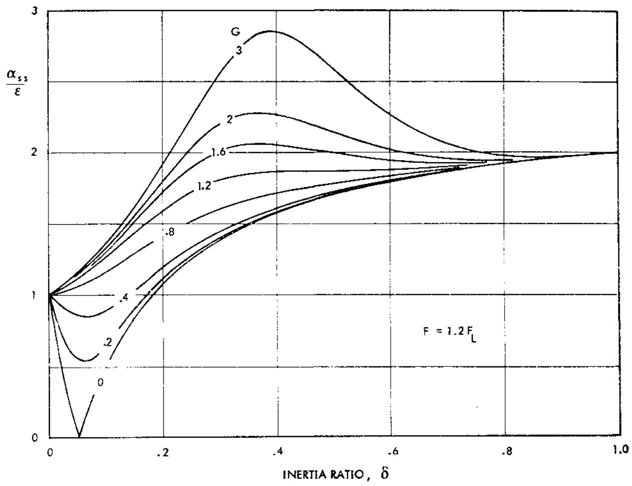

Fig. 5 Performance, $F = 1.2\ F_L$.

be shown subsequently, that a decrease in steady-state amplitude is accompanied by an increase in damping time constant.

Convergence and Minimum Response

A practical system must combine good acquisition performance with good steady-state performance. A measure of a system's capability to acquire is the damping time of the least damped mode. If $-D_1$ is the real part of the root of the characteristic equation closest to the imaginary axis the time to damp to $1/e$ of the initial amplitude is shown in Ref. 1 to be given by

$$t_e/T = \omega t/2\pi\tau D_1 = 0.09188/D_1 p_1^{1/2} \quad (46)$$

or, if the $p_1 = 1$ assumption is adopted

$$t_e = 0.09188/D_1 \text{ orbits} \quad (47)$$

If the relative importance of steady-state response and damping time can be specified, a performance criterion is expressible in the form

$$\Phi = t_e + k\alpha_{ss}/\epsilon \quad (48)$$

Minimization of Φ would then lead to the definition of the parameters for an optimum system. Unfortunately, the value of k in Eq. (48) is difficult to establish.

Two alternative approaches which may be used as rational design guides are 1) Minimize the damping time for a chosen value of α_{ss}/ϵ and 2) minimize α_{ss}/ϵ for a chosen level of damping. It will be easier, in general, to chose α_{ss}/ϵ or t_e and use 1 or 2 than to estimate a value of k in (48).

It is not possible to express D_1 analytically, thus the relation between damping time and steady state amplitude must be approached indirectly.

The following procedure was used initially:

1) For a given α_{ss}/ϵ, δ, and F, find G from (33), i.e.,

$$G^2/3 = \{(\alpha_{ss}/\epsilon)^2[F(1 - 2\delta) - \tfrac{4}{3}]^2 - [F(1 + \delta) - \tfrac{4}{3}]^2\}/\{(1 + \delta)^2 - (\alpha_{ss}/\epsilon)^2(1 - 2\delta)^2\} \quad (49)$$

2) Construct the characteristic equation in terms of δ, K, and the values of G, in terms of δ and K from (49)

$$(C.E) = s^4 + (1 + \delta)Gs^3 + K(1 + \delta)s^2/(1 - \delta) + (1 - \delta)Gs + K - 1 \quad (50)$$

$$K = F/F_L = F/(1 - \delta)$$

3) Factor the characteristic equation to obtain value of D_i.

4) Pick D_1, i.e., the maximum value of D_i.

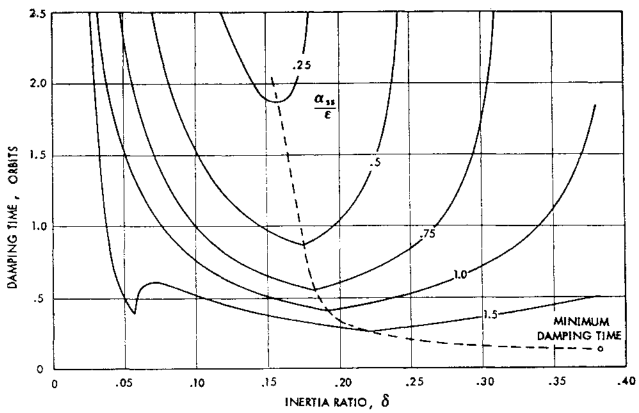

Fig. 6 Design chart.

5) Plot D_1 against K for various value of δ and find the maximum (corresponding to minimum t_e).

The resulting values of t_e minimum can then be presented as a function of δ for a given α_{ss}/ϵ. Curves for several values of α_{ss}/ϵ are shown in Fig. 6. The procedure outlined earlier proved very tedious and time consuming. It appeared worthwhile to modify an existing algebraic optimization program[4] to perform the steps digitally. This was done and proved successful. The digital results confirmed the graphically obtained points including the unexpected secondary minimum that occurs in the $\alpha_{ss}/\epsilon = 1.5$ curve of Fig. 6.

A trade curve, showing the relation between minimum damping time and minimum steady-state response, is shown in Fig. 7. This curve is a crossplot of the dotted line in Fig. 6. It shows the region of realizable performance and indicates clearly the conflict between the two desirable performance characteristics. To the left, the curve tends to an infinite damping time as the steady-state response tends to zero. To the right, the curve would bottom out with $t_e = 0.137$ at $\alpha_{ss}/\epsilon = 2.245$, the point corresponding to the maximum damping of the least damped mode as determined in Ref. 1.

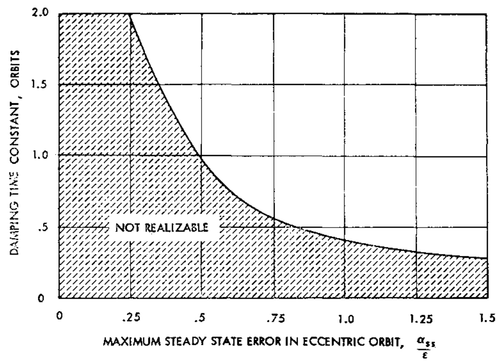

Fig. 7 Trade between steady-state error and transient damping time.

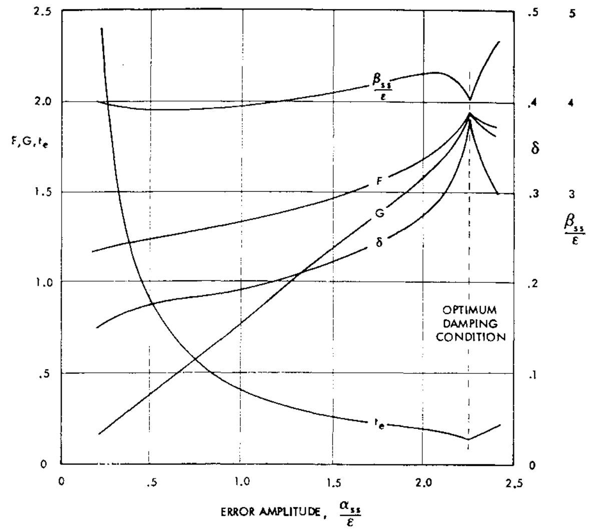

Fig. 8 Optimum design chart.

The results are summarized in the design chart of Fig. 8 which shows β_{ss}/ϵ, δ, F, G, and t_e as functions of α_{ss}/ϵ for the locus of best designs.

Conclusions

An analysis has been made of the response of a gravity gradient system with a single damping boom in an eccentric orbit. The effect of small eccentricity was approximated by a sinusoidal disturbance torque in the orbit plane and the conditions for minimum steady state response examined. It was found that the steady-state error could be reduced by decreasing the damping and could be made to approach zero for certain configurations. In general the conditions for good response and good samping were found to be incompatible and indicated that a compromise must be made in a practical system. No over-all measure of system performance seems possible and the choice of system parameters depends on the relative importance of damping time and steady-state error. Curves are presented to aid the designer in making this choice.

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